

Chow-Künneth decomposition for 3- and 4-folds fibred by varieties with small Chow group of zero-cycles

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Abstract

Let $f : X \rightarrow S$ be a dominant morphism from a smooth projective variety X to a smooth projective variety S of dimension ≤ 2 over a field k with general fibre having trivial Chow group of zero-cycles. For example, X could be the total space of a two-dimensional family of varieties whose general member is rationally connected. Suppose that X has dimension ≤ 4 . Then, we prove that X has a self-dual Murre decomposition. Moreover we prove that the motivic Lefschetz conjecture holds for X and hence so does the Lefschetz standard conjecture. We also give new examples of threefolds of general type which are Kimura finite dimensional, new examples of fourfolds of general type having a self-dual Murre decomposition, as well as new examples of varieties with finite degree three unramified cohomology.

Introduction

Throughout this paper, algebraic cycles and Chow groups are with rational coefficients. In characteristic zero the Hodge conjecture predicts that every Hodge class in $H_{2i}(X) := H_{2i}(X(\mathbf{C}), \mathbf{Q})$ is the class of an algebraic cycle. In particular, given any two smooth projective varieties X and Y over k , the Hodge conjecture predicts that any morphism f of Hodge structures between $H_*(X)$ and $H_*(Y)$ comes from geometry. By this we mean that f is induced by a correspondence between X and Y , that is by an algebraic cycle on $X \times Y$. Whether the Hodge conjecture happens to be true or not, Grothendieck pointed out that certain morphisms of Hodge structure play a more important role in the theory of algebraic cycles. If X has pure dimension d , he suggested that the Künneth component in $H_{2d}(X \times X)$ inducing the projector on $H_i(X)$ should be induced by a correspondence. He also suggested that the inverse to the hard Lefschetz isomorphism $H_{2d-i}(X) \rightarrow H_i(X)$ given by intersecting $d - i$ times with the class of a smooth hyperplane section should be induced by an algebraic cycle. The first conjecture is usually referred to as the Künneth standard conjecture and the second one to the Lefschetz standard conjecture. Classically, it is known that the latter is stronger than the former [13, 4.1].

If the Hodge conjecture gives a simple description of the image of the cycle class map $CH_i(X) \rightarrow H_{2i}(X)$ in terms of the Hodge structure of $H_{2i}(X)$, it is a much more difficult

problem to unravel the nature of the kernel of the cycle class map. Beilinson and Bloch, inspired by Grothendieck's philosophy of motives, first proposed a description of such a kernel in terms of a descending filtration on Chow groups that would behave functorially with respect to the action of correspondences and would be such that its graded parts would depend solely on the topology of X .

Twenty years ago Murre proposed that not only should the Künneth projectors in cohomology be induced by correspondences, but also they should be induced by correspondences that are idempotents modulo rational equivalence. In [17], Murre conjectured the following.

- (A) X has a Chow-Künneth decomposition $\{\pi_0, \dots, \pi_{2d}\}$: There exist mutually orthogonal idempotents $\pi_0, \dots, \pi_{2d} \in CH_d(X \times X)$ adding to the identity such that $(\pi_i)_* H_*(X) = H_i(X)$ for all i .
- (B) $\pi_0, \dots, \pi_{2l-1}, \pi_{d+l+1}, \dots, \pi_{2d}$ act trivially on $CH_l(X)$ for all l .
- (C) $F^i CH_l(X) := \text{Ker}(\pi_{2l}) \cap \dots \cap \text{Ker}(\pi_{2l+i-1})$ doesn't depend on the choice of the π_j 's. Here the π_j 's are acting on $CH_l(X)$.
- (D) $F^1 CH_l(X) = CH_l(X)_{\text{hom}}$, where the subscript hom refers to homologically trivial cycles.

A variety X that satisfies all four conjectures is said to have a *Murre decomposition*. If moreover the Chow-Künneth decomposition of conjecture (A) can be chosen so that $\pi_i = {}^t \pi_{2d-i} \in CH_d(X \times X)$, then X is said to have a *self-dual Murre decomposition*.

The relevance of Murre's conjectures was demonstrated by Jannsen [9] who showed that these hold for all smooth projective varieties if and only if Bloch's and Beilinson's conjectures hold for all smooth projective varieties. Murre's formulation of a conjectural descending filtration on Chow groups has the advantage over Beilinson's and Bloch's that it does not involve any functoriality properties and that it can therefore be proven on a case by case basis. Since Murre's paper [17] appeared, many authors have tried to prove those conjectures for certain classes of varieties. In this paper we extend the list of cases for which these can be proven.

Let X be a smooth projective variety defined over a field k and let Ω be a universal domain over k that contains $k(X)$, i.e. Ω is an algebraically closed field of infinite transcendence degree over k that contains $k(X)$. A smooth projective variety X will be said to have trivial Chow group of zero-cycles if $CH_0(X_\Omega) = \mathbf{Q}$. Our main result is the following

Theorem 1. *Let $f : X \rightarrow S$ be a dominant morphism between smooth projective varieties defined over a field k such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Suppose that S has dimension ≤ 2 and that X has dimension ≤ 4 . Then, X has a self-dual Murre decomposition. Moreover the motivic Lefschetz conjecture, as stated in §5, holds for X and hence so does the Lefschetz standard conjecture.*

Let's stress that the theorem gives a self-dual Murre decomposition of X which is defined over a field of definition of f .

This result contrasts with the approach of Gordon-Hanamura-Murre [7] where Chow-Künneth decompositions are constructed for varieties X that come with a fibration $f : X \rightarrow S$ which is “nice” enough: it is assumed among other things in *loc. cit.* that f should be smooth away from a finite number of points on S and that f should have a relative Chow-Künneth decomposition. Here we do not even require f to be flat.

Theorem 1 was already proved in the case $\dim S \leq 1$: a self-dual Chow-Künneth decomposition for which the motivic Lefschetz conjecture holds was constructed in [21, 4.6] and Murre’s conjectures were checked to hold in [20, 4.21]. The results in [20, 21] just mentioned are more generally valid for fourfolds with Chow group of zero-cycles supported on a curve. By theorem 1.3, it is the case that if X is as in theorem 1 with $\dim S \leq 1$ that $CH_0(X_\Omega)$ is supported on a curve. From now on, we will therefore focus on the case $\dim S = 2$.

Here, Murre’s conjecture (C) is proved only on the grounds that the idempotents of a Chow-Künneth decomposition for X are supported in a specific dimension, cf. section 6. It is however proved under some extra assumption on X in proposition 6.3 and theorem 8.3. Let’s mention that del Angel and Müller-Stach [4] proved the existence of a Murre decomposition for threefolds fibred by conics over a surface (see also the recent paper [15] where Chow-Künneth decompositions are constructed for some threefolds including conic fibrations). In addition to treating the four-dimensional case, our theorem makes more precise the result of [4] by showing that the Murre decomposition can be chosen to be self-dual and by showing the motivic Lefschetz conjecture for X . Also our approach is different from [4]. Del Angel and Müller-Stach assume that all the fibres of f are rationally connected, this allows them to compute the cohomology of X via the Leray spectral sequence. They then construct idempotents modulo rational equivalence and check that they act as the Künneth projectors on cohomology. Here, we do not make any assumptions on the bad fibres of f and we first compute the Chow group of zero-cycles of X to only then deduce that the idempotents we construct act as the Künneth projectors on cohomology.

A word about the proof of the theorem and about the organisation of the paper. In section 1 we make the simple observation that $f_* : CH_0(X_\Omega) \rightarrow CH_0(S_\Omega)$ is bijective with inverse induced by an algebraic correspondence which is defined over a field of definition of f . Together with theorem 2.1, the proof of the existence of a Murre decomposition for X essentially reduces to the case of motives of surfaces. The validity of Murre’s conjectures for surfaces goes back to Murre himself [16]. However, by the very nature of theorem 2.1, we need Murre’s conjectures not only for surfaces but for motives of surfaces (i.e. we need to deal with idempotents). This is the object of section 3. The construction of idempotents inducing the right Künneth projectors for X in homology is carried out in section 4. Constructing such idempotents is easy from the case of surfaces. However these are not necessarily mutually orthogonal. The non-commutative Gram-Schmidt process which already appears in [21] and which is run on this set of idempotents is described in lemma 4.2. This way we obtain a self-dual Chow-Künneth decomposition for X . The

motivic Lefschetz conjecture is formulated in section 5, its relevance is discussed and it is proved for X there. Murre's conjectures are then proved for X in section 6 by using the results of [20]. If C is a smooth projective curve, the results of *loc. cit.* actually make it possible to prove parts of Murre's conjectures for $X \times C$. This is the object of section 7.

Murre's observation that the Künneth projectors should lift to idempotents modulo rational equivalence is crucial in the sense that a combination of Beilinson's and Bloch's conjectures with Grothendieck's standard conjectures imply that any projector in homology should be liftable to an idempotent modulo rational equivalence. Kimura [12] introduced a notion of finite dimensionality for Chow motives which implies such a lifting property for projectors. Kimura's notion of finite dimensionality has become widely popular for this reason and more importantly because of its relationship to Murre's conjectures. The simple observation of theorem 1.3 is used in section 8 to give new examples of threefolds of general type which are Kimura finite-dimensional, namely threefolds fibred by Godeaux surfaces. There, using the result of section 4 we also show in theorem 8.2 that if X is a conic fibration over a surface which is Kimura finite-dimensional, then X is Kimura finite-dimensional. In section 9, we produce examples of fourfolds of general type with Chow group of zero-cycles not supported on a curve but which admit a self-dual Murre decomposition. Such examples will be given by fourfolds fibred by surfaces birational to Godeaux surfaces. Theorem 1.3 is slightly generalised in theorem 1.7 ; this is used in section 10 to prove finiteness of unramified cohomology in some new cases.

Finally, although we don't state it here, the methods of this paper actually show that Murre's conjectures hold for smooth projective fourfolds X such that there exist a smooth projective surface S and correspondences $\alpha \in CH_2(S \times X)$ and $\beta \in CH^2(X \times S)$ such that $\beta \circ \alpha = \Delta_S \in CH_2(S \times S)$ and such that $\alpha \circ \beta$ acts as the identity on $CH_0(X_\Omega)$. Consequently, if $f : X \rightarrow S$ is a dominant morphism to a surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles, then Murre's conjectures hold for any smooth projective variety X' which is birational to X .

Notations. We refer to [19] for the notion of Chow motive and to [11] for the covariant notations we use here. Briefly, a Chow motive M is a triple (X, p, n) where X is a smooth projective variety over k of pure dimension d , $p \in CH_d(X \times X)$ is an idempotent ($p \circ p = p$) and n is an integer. The motive M is said to be effective if $n \geq 0$. A morphism between two motives (X, p, n) and (Y, q, m) is a correspondence in $q \circ CH_{d+m-n}(X \times Y) \circ p$. We write $\mathfrak{h}(X)$ for the motive of X , i.e. for the motive $(X, \Delta_X, 0)$ where Δ_X is the class of the diagonal inside $CH_d(X \times X)$. We have $CH_i(X, p, n) = p_* CH_{i-n}(X)$ and $H_i(X, p, n) = p_* H_{i-2n}(X)$, where we write $H_i(X) := H^{2d-i}(X_{\bar{k}}, \mathbf{Q}_\ell)$ for ℓ -adic homology.

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1 A geometric result on zero-cycles

Let X and S be smooth projective varieties over k and let $f : X \rightarrow S$ be a dominant morphism. Then, there exists a general linear section $\sigma : H \rightarrow X$ of dimension $\dim S$ which is smooth over k and such that the morphism $\pi := f|_H : H \rightarrow S$ is dominant. In particular π is generically finite and its degree is written n .

Proposition 1.1. *Let $f : X \rightarrow S$ be a dominant morphism. With the notations above, $\Gamma_f \circ \Gamma_\sigma \circ {}^t\Gamma_\pi = n \cdot \Delta_S \in CH_{\dim S}(S \times S)$. In particular $f_* : CH_0(X) \rightarrow CH_0(S)$ is surjective.*

Proof. This follows from the projection formula applied to $\pi = f \circ \sigma$. \square

Definition 1.2. Let $f : X \rightarrow S$ be a dominant morphism between smooth projective varieties defined over a field k . A general point of S is a closed point sitting outside a given proper closed subset of S . By *general fibre* of f , we mean the fibre of f over a general point of S .

Theorem 1.3. *Let $f : X \rightarrow S$ be a dominant morphism between smooth projective varieties defined over a field k . Assume that a general fibre Y of f satisfies $CH_0(Y) = \mathbf{Q}$. Then, the induced map $f_* : CH_0(X) \rightarrow CH_0(S)$ is bijective and its inverse is induced by a correspondence $\Gamma \in CH^{\dim X}(S \times X)$. Moreover, Γ can be chosen to be defined over a field of definition of f .*

Proof. Let's show that the correspondence Γ can be chosen to be $\frac{1}{n}\Gamma_\sigma \circ {}^t\Gamma_\pi$. According to proposition 1.1 it suffices to prove that the correspondence $\Gamma_\sigma \circ {}^t\Gamma_\pi \circ \Gamma_f$ acts as multiplication by n on $CH_0(X)$.

Let's fix an open subset U of S such that $\pi : H_U \rightarrow U$ is finite and such that the fibres of f_U satisfy $CH_0(X_u) = \mathbf{Q}$ for all closed points u in U .

Let p be a closed point of X and let's show that $\sigma_* \pi^* f_*[p] = n \cdot [p] \in CH_0(X)$ where $[p]$ denotes the class of p in $CH_0(X)$. By Chow's moving lemma, the zero-cycle $[p] \in Z_0(X)$ is rationally equivalent to a zero-cycle $\alpha = \sum a_i \cdot [p_i]$ supported on X_U . This means that each p_i is a closed point of X that belongs to the open subset X_U of X . Let $u_i := f(p_i)$. By proper push-forward we have a map $f_* : CH_0(X) \rightarrow CH_0(S)$ with $f_*[p] = f_*(\sum a_i \cdot [p_i]) = \sum a_i \cdot [k(p_i) : k(u_i)] \cdot [u_i]$. Let $v_{i,1}, \dots, v_{i,n_i}$ be the points in the preimage $\pi^{-1}(u_i)$ counted with multiplicity. Because $\pi : H_U \rightarrow U$ is finite of degree n , we have the formula $\sum_j [k(v_{i,j}) : k] = n \cdot [k(u_i) : k]$. The morphism $\pi : H \rightarrow S$ is not necessarily flat and we therefore cannot conclude that $\pi^*[s] = [\pi^{-1}(s)]$ for a closed point $s \in S$. However, because H and S are smooth over k , we can compute $\pi^*[s]$ as being $(p_H)_*({}^t\Gamma_\pi \cap (p_S)^*[s])$ where $p_H : S \times H \rightarrow H$ is projection on H and p_S is projection on S . Now by choice of the open $U \subset S$ and of the u_i 's, we have that ${}^t\Gamma_\pi$ intersects $[u_i \times H]$ properly for all i . Therefore we get $\pi^* f_*[p] = \sum a_i \cdot [k(p_i) : k(u_i)] \cdot [v_{i,j}]$.

Because $\sigma : H \rightarrow X$ is a closed embedding, we have for any closed point h in H , $\sigma_*[h] = [\sigma(h)]$. It follows that $\sigma_*\pi^*f_*[p] = \sum a_i \cdot [k(p_i) : k(u_i)] \cdot [\sigma(v_{i,j})] \in CH_0(X)$. Now, $CH_0(X_u) = \mathbf{Q}$ for all $u \in U$. Because $f \circ \sigma = \pi$, the closed points p_i and $\sigma(v_{i,j})$ belong to the same fibre X_{u_i} . Therefore, by assumption on $CH_0(X_{u_i})$ (X_{u_i} is a general fibre), the zero-cycle $[p_i]$ is rationally equivalent to $\frac{\deg_k(p_i)}{\deg_k(v_{i,j})} \cdot [\sigma(v_{i,j})]$ for all j . This yields $\sigma_*\pi^*f_*[p] = \sum a_i \cdot \frac{\deg_k(v_{i,j})}{\deg_k(u_i)} \cdot [p_i] = n \sum a_i \cdot [p_i] = n \cdot [p] \in CH_0(X)$. \square

For example, we get as a corollary the following which is used in [20, Cor. 4.23] and [21, Cor. 4.7] in the cases when S is a curve.

Corollary 1.4. *Let $f : X \rightarrow S$ be a dominant morphism between smooth projective varieties defined over a field k . Assume that the general fibre of f is rationally connected (e.g. X could be a Fano fibration). Then, $f_* : CH_0(X) \rightarrow CH_0(S)$ is bijective and there is a correspondence $\Gamma \in CH^{\dim X}(S \times X)$ such that $\Gamma_* : CH_0(S) \rightarrow CH_0(X)$ is the inverse of f_* .* \square

Theorem 1.7 below, which generalises theorem 1.3 is irrelevant to the proof of theorem 1. However, we include it here because of the interesting consequences it has for unramified cohomology, see section 10.

Definition 1.5. A smooth projective variety X over k is said to have *representable* Chow group of algebraically trivial i -cycles if there exists a curve C over Ω and a correspondence $\gamma \in CH_{i+1}(C \times X_\Omega)$ such that $\gamma_*CH_0(C)_{alg} = CH_i(X_\Omega)_{alg}$.

Lemma 1.6. *Let X be a smooth projective variety over k . Then, the following statements are equivalent.*

1. $CH_0(X)_{alg}$ is representable.
2. The Albanese map $\text{alb}_{X_\Omega} : CH_0(X_\Omega)_{alg} \rightarrow \text{Alb}_{X_\Omega}(\Omega)$ is an isomorphism (this map is always surjective).
3. If $\iota : C \rightarrow X$ is any smooth linear section of X of dimension 1, the induced map $\iota_* : CH_0(C_\Omega) \rightarrow CH_0(X_\Omega)$ is surjective.
4. If $\iota : C \rightarrow X$ is any smooth linear section of X of dimension 1, the induced map $\iota_* : CH_0(C_K) \rightarrow CH_0(X_K)$ is surjective for all field extensions K/k .

Proof. The first three statements are equivalent by [9, 1.6] and the fourth statement clearly implies the third statement. Let's thus consider a smooth linear section $\iota : C \rightarrow X$ of dimension 1. Assume $\iota_* : CH_0(C_\Omega) \rightarrow CH_0(X_\Omega)$ is surjective. A decomposition of the diagonal as performed in [1] shows that $\Delta_X = \Gamma_1 + \Gamma_2$ with Γ_1 supported on $D \times X$ for some divisor D and Γ_2 supported on $X \times C$. By Chow's moving lemma Γ_1 acts trivially on $CH_0(X_K)$ for all field extensions K/k . Therefore $CH_0(X_K) = (\Gamma_2)_*CH_0(X_K)$, and hence $\iota_*CH_0(C_K) = CH_0(X_K)$ for all field extensions K/k . \square

Theorem 1.7. *Let $f : X \rightarrow S$ be a generically smooth and dominant morphism between smooth projective varieties defined over a field k . Assume that the general fibre Y of f is such that $CH_0(Y)_{\text{alg}}$ is representable. Then, $CH_0(X)$ is supported in dimension $\dim S + 1$. This means that there exists a smooth projective variety T over k of dimension $\dim S + 1$ and a correspondence $\Gamma \in CH^{\dim X}(T \times X)$ such that $(\Gamma_K)_* : CH_0(T_K) \rightarrow CH_0(X_K)$ is surjective for all field extensions K/k .*

Proof. Let $\iota : H \rightarrow X$ be a smooth linear section of X of dimension $\dim S + 1$ such that f restricted to H is dominant and generically smooth.

Let U be an open subset of S such that $f_U : X_U \rightarrow U$ is smooth, $f_U|_{H_U} : H_U \rightarrow U$ is smooth and such that the fibres of f_U have representable Chow group of zero-cycles.

Up to base-change together with lemma 1.6, it is enough to prove that $\iota_* : CH_0(H) \rightarrow CH_0(X)$ is surjective. Let p be a closed point of X . By Chow's moving lemma, the zero-cycle $[p]$ is rationally equivalent to a cycle $\alpha = \sum a_i \cdot [p_i]$ supported on X_U . Let $s_i := f(p_i) \in U$. Then, by choice of U , each cycle $[p_i]$ is rationally equivalent on X_{s_i} to a cycle β_i supported on H_{s_i} . Now clearly $\sum a_i \cdot \beta_i$ is in the image of $\iota_* : CH_0(H) \rightarrow CH_0(X)$ and hence so is $[p]$. \square

Remark 1.8. The descent properties of theorems 1.3 & 1.7, i.e. the fact that the correspondence Γ in those theorems can be chosen to be defined over a field of definition of f , are essential to proving that X has a Chow-Künneth decomposition defined over the field of definition of f (theorem 4.1) and to proving proposition 10.1.

Remark 1.9. Under the assumptions of the above theorem, it is not true that if $CH_0(S)$ is supported in dimension, say n , that $CH_0(X)$ is supported in dimension $n + 1$. Consider for example a Lefschetz fibration $S \rightarrow \mathbf{P}^1$ associated to a smooth projective surface S with non-representable Chow group of zero-cycles.

Remark 1.10. If one believes in the Bloch-Beilinson conjectures, the above theorem can be extended to the following. Let $f : X \rightarrow S$ be a dominant morphism between smooth projective varieties defined over an algebraically closed field k . Let n be a positive integer and assume that the general fibre Y of f is such that $CH_0(Y)$ is supported in dimension n . Then, $CH_0(X)$ is supported in dimension $\dim S + n$.

2 Effective motives with trivial Chow group of zero-cycles

The following theorem is a consequence of the technique of Bloch and Srinivas [1]. It appears in Kahn-Sujatha [10, 2.4.1].

Theorem 2.1. *Let $M = (X, p)$ be an effective Chow motive such that $CH_0(M_\Omega) = 0$. Then, there exists a smooth projective variety Y of dimension $\dim X - 1$ and an idempotent $q \in CH_{\dim Y}(Y \times Y)$ such that $(X, p, 0) \simeq (Y, q, 1)$.*

Proof. Without loss of generality, we can assume that X is connected. Since we are working with rational coefficients, the assumption $CH_0(M_\Omega) = 0$ implies $CH_0(M_{k(X)}) = 0$ which means $p_*CH_0(X_{k(X)}) = 0$. In particular, if η_X denotes the generic point of X , we have $p_*\eta_X = 0$. But, $p_*\eta_X$ is the restriction of $p \in CH_d(X \times X)$ to $\varinjlim CH_d(U \times X) = CH_0(X_{k(X)})$ where the limit is taken over all open subsets U of X . Therefore, by the localization exact sequence for Chow groups, there exists a proper closed subset $D \subset X$ and a correspondence $\gamma \in CH_d(D \times X)$ such that γ maps to p via the inclusion $D \times X \rightarrow X \times X$. Up to shrinking the open U for which $p|_{U \times X}$ vanishes, we can assume that D has pure dimension $d - 1$. Let $Y \rightarrow D$ be an alteration of D and let $\sigma : Y \rightarrow D \hookrightarrow X$ be the composite morphism. The induced map $CH_d(Y \times X) \rightarrow CH_d(D \times X)$ is surjective and we have $p = (\sigma \times \text{id}_X)_*f$, where $f \in CH_d(Y \times X)$ is a lift of γ . Then, by [5, 16.1.1], we have $(\sigma \times \text{id}_X)_*f = f \circ {}^t\Gamma_\sigma$. This yields a factorisation $p = f \circ g$, where $f \in CH_d(Y \times X)$ and $g = {}^t\Gamma_\sigma \in CH^d(X \times Y)$. Let's consider the correspondence $q := g \circ f \circ g \circ f = g \circ p \circ f \in CH_{d-1}(Y \times Y)$. It is straightforward to check that q is an idempotent, and that $p \circ f \circ q \circ g \circ p = p$ as well as $q \circ g \circ p \circ f \circ q = q$. These last two equalities exactly mean that $p \circ f \circ q$ seen as a morphism of Chow motives from $(Y, q, 1)$ to $(X, p, 0)$ is an isomorphism with inverse $q \circ g \circ p$. \square

3 The Albanese motive and the Picard motive

As the previous section shows, it is convenient not only to deal with smooth projective varieties but also with idempotents. It is however often difficult to deal with idempotents because they are usually not central. Here we extend the construction of Murre's Albanese projector to the case of Chow motives.

We need a basic lemma.

Lemma 3.1. *Let \mathcal{C} be a semi-simple abelian category and let A and B be two objects of \mathcal{C} . Let also $f \in \text{Hom}_{\mathcal{C}}(A, B)$ be a morphism between A and B . Then, there is a morphism $g \in \text{Hom}_{\mathcal{C}}(B, A)$ such that $f \circ g$ is an idempotent with image $\text{Im } f$.*

Proof. Since the category \mathcal{C} is assumed to be semi-simple, there exist objects M and N such that $A = \text{Ker } f \oplus M$ and $B = N \oplus \text{Im } f$. The map $f|_M : M \rightarrow B$ induces an isomorphism on its image $\text{Im } f$ that we write $f' : M \rightarrow \text{Im } f$. Let's now define a map $g : B \rightarrow A$ as $g := \begin{pmatrix} 0 & 0 \\ 0 & f'^{-1} \end{pmatrix} : N \oplus \text{Im } f \rightarrow \text{Ker } f \oplus M$. Then, clearly, $f \circ g$ is an idempotent with image $\text{Im } f$. \square

We now fix a Chow motive (X, p) with X of pure dimension d over a field k . Without loss of generality, we can suppose that X is connected. We also fix $\iota : C \rightarrow X$ a smooth linear section of X of dimension one and we let z be a zero-cycle on C of degree one. The zero-cycle $y := \iota_*z \in CH_0(X)$ has positive degree.

Although the following two propositions are obvious, we write them down because they contain all the ingredients needed for the construction of the Albanese projector and the Picard projector.

Proposition 3.2. *Let $M = (X, p)$ be an effective Chow motive. Then, there exists an idempotent $p_0 \in p \circ CH_d(X \times X) \circ p$ (i.e. (X, p_0) is a direct summand of (X, p)) such that $(p_0)_* H_*(X) = p_* H_0(X)$ and such that $(X, p_0) \simeq \mathbb{1}^n$ with $n = 0$ or 1 .*

Proof. Let $P := \text{Spec } k$ be a point and let $\Gamma \in CH_0(P \times X)$ be a correspondence such that $\Gamma_* H_0(P) = H_0(X)$. Then $p \circ \Gamma$ induces a map $H_0(P) \rightarrow H_0(X)$ with image $p_* H_0(X)$. Since the category of vector spaces is abelian semi-simple, there exists $\alpha \in \text{Hom}(H_0(X), H_0(P))$ such that $p \circ \Gamma \circ \alpha$ is an idempotent with image $\text{Im}(p \circ \Gamma)_*$. Clearly, α is induced by a correspondence $A \in CH_d(X \times P)$. We set $p_0 := p \circ \Gamma \circ A \circ p$. Clearly we have $p \circ p_0 \circ p = p_0$. The relation $\alpha \circ (p \circ \Gamma)_* \circ \alpha = \alpha$ immediately implies that $A \circ p \circ \Gamma \circ A = A$. Hence p_0 is an idempotent. Let's define $\pi := A \circ p \circ \Gamma \circ A \circ p \circ \Gamma$. This is an idempotent in $CH_0(P \times P)$ as can be immediately checked. Now, it is easy to see that the correspondence $p \circ \Gamma \circ \pi$ induces an isomorphism of Chow motives $(P, \pi) \rightarrow (X, p_0)$ whose inverse is given by $\pi \circ A \circ p$. It is then easy to see that $(X, p_0) \simeq \mathbb{1}^n$ with $n = 0$ or 1 . \square

Proposition 3.3. *Let $M = (X, p)$ be an effective Chow motive. Then, there exists an idempotent $p_{2d} \in p \circ CH_d(X \times X) \circ p$ such that $(p_{2d})_* H_*(X) = p_* H_{2d}(X)$ and such that $(X, p_{2d}) \simeq \mathbb{1}(d)^{\oplus n}$ with $n = 0$ or 1 .*

Proof. Let q be the projector constructed in the previous proposition for ${}^t p$. Then, $p_{2d} := {}^t q$ is the required projector. \square

Let's recall that the Albanese map $\text{alb}_X : CH_0(X) \rightarrow \text{Alb}_X(k)$ is functorial with respect to the action of correspondences [18, (1.4)-(1.7)]. Precisely, if $\gamma \in CH_{\dim X}(X \times Y)$, then there are induced maps $\gamma_* : CH_0(X) \rightarrow CH_0(Y)$ and $\gamma_* : \text{Alb}_X(k) \rightarrow \text{Alb}_Y(k)$ satisfying $\gamma_* \circ \text{alb}_X = \text{alb}_Y \circ \gamma_*$. Thus it makes sense to speak of the Albanese map for a motive $M = (X, p)$.

Proposition 3.4 (cf. th. 3.9 and prop. 3.10 of [19]). *Let Y and Z be connected smooth projective varieties over k and let $\zeta \in CH_0(Y)$ and $\eta \in CH_0(Z)$ be 0-cycles of positive degree. Then, there is a functorial isomorphism*

$$\Omega : \text{Hom}_k(\text{Alb}_Y, \text{Pic}_Z^0) \otimes \mathbf{Q} \xrightarrow{\simeq} \{c \in CH^1(Y \times Z) \mid c(\zeta) = {}^t c(\eta) = 0\}.$$

Theorem 3.5. *Let $M = (X, p)$ be an effective Chow motive. Then, there exists an idempotent $p_1 \in p \circ CH_d(X \times X) \circ p$ (i.e. (X, p_1) is a direct summand of (X, p)) such that $(p_1)_* H_*(X) = p_* H_1(X)$ and such that $(X, p_1) \simeq \mathfrak{h}_1(p_* \text{Alb}_X)$. The idempotent p_1 is called the Albanese projector.*

Proof. Recall that we are given $\iota : C \rightarrow X$ a smooth linear section of X of dimension one and a zero-cycle z on C of degree one. The weak Lefschetz theorem says that the induced map $\iota_* : H_1(C) \rightarrow H_1(X)$ is surjective. After composing the correspondence $\Gamma_\iota \in CH_1(C \times X)$ with $\Delta_C - [z \times C] - [C \times z] \in CH_1(C \times C)$, we get a correspondence $\gamma \in CH_1(C \times X)$ such that $\gamma_* H_*(C) = H_1(X)$, $\gamma_* : \text{Alb}_C \rightarrow \text{Alb}_X$ is surjective and such that $\gamma_* z = 0$. Let's then define $\Gamma := p \circ \gamma = p \circ \Gamma_\iota \circ (\Delta_C - [z \times C] - [C \times z])$.

It is well-known that the category of abelian varieties over k up to isogeny is semi-simple. By functoriality, the idempotent $p \in CH_d(X \times X)$ acts on Alb_X and decomposes Alb_X as a direct sum

$$\text{Alb}_X = \text{Im } p_* \oplus \text{Ker } p_*$$

where $\text{Im } p_* = \text{Im } (p_* : \text{Alb}_X \rightarrow \text{Alb}_X)$ and $\text{Ker } p_* = \text{Ker } (p_* : \text{Alb}_X \rightarrow \text{Alb}_X)$.

The correspondence Γ induces a map $\Gamma_* : \text{Alb}_C \rightarrow \text{Alb}_X$ with image $\text{Im } p_*$. Thanks to lemma 3.1 and its proof, there is a morphism $\alpha \in \text{Hom}_k(\text{Alb}_X, \text{Alb}_C) \otimes \mathbf{Q}$ such that $\alpha|_{\text{Ker } p_*} = 0$ and such that $\Gamma_* \circ \alpha$ is an idempotent with image $\text{Im } p_*$.

After having identified Alb_C with Pic_C^0 , we see thanks to proposition 3.4 that the morphism α is induced by a correspondence $A \in CH^1(X \times C)$ with respect to the choice of the positive degree zero-cycle z on C and to the positive degree zero-cycle $\iota_* z$ on X .

Clearly the correspondence

$$p_1 := \Gamma \circ A \circ p \in CH_d(X \times X)$$

satisfies $p \circ p_1 \circ p = p_1$ as well as $(p_1)_* \text{Alb}_X = p_* \text{Alb}_X$. We claim that p_1 is an idempotent. In order to show this, it is enough to see by proposition 3.4 that

$$\alpha \circ \Gamma_* \circ \alpha = \alpha \in \text{Hom}(\text{Alb}_X, \text{Alb}_C) \otimes \mathbf{Q},$$

which is straightforward from the choice of α .

Let's prove that $(X, p_1) \simeq \mathfrak{h}_1(p_* \text{Alb}_X)$. For this purpose let's define $\pi := A \circ \Gamma \in CH^1(C \times C)$. Because $\pi_* z = \pi^* z = 0$, by proposition 3.4 we can check that π is an idempotent by simply checking that it acts as an idempotent on Alb_C . We could also have defined $\pi := A \circ \Gamma \circ A \circ \Gamma$, which is clearly an idempotent since $\Gamma \circ A \circ p$ is.

On the one hand, we have an isomorphism of Chow motives $(C, \pi) \simeq \mathfrak{h}_1(p_* \text{Alb}_X)$. Indeed, the idempotent π satisfies $\pi_* z = \pi^* z = 0$ and its action π_* on Alb_C factors through the map $(\Gamma \circ A \circ \Gamma)_* : \text{Alb}_C \rightarrow \text{Alb}_X$ (whose image is $p_* \text{Alb}_X$) followed by the injective map $(A_*)|_{p_* \text{Alb}_X} : p_* \text{Alb}_X \rightarrow \text{Alb}_C$.

On the other hand, it is straightforward to check that $p_1 \circ \Gamma \circ \pi \in \text{Hom}((C, \pi), (X, p_1))$ is an isomorphism of Chow motives whose inverse is $\pi \circ A \circ p_1$.

Finally, the idempotent p_1 induces the required Künneth projector, i.e. $(p_1)_* H_*(X) = p_* H_1(X)$. Indeed, we have $(p_1)_* H_*(X) = H_*(X, p_1) \simeq H_*(\mathfrak{h}_1(p_* \text{Alb}_X)) = H_1(p_* \text{Alb}_X)$ and so $(p_1)_* H_*(X)$ has weight 1, i.e. $(p_1)_* H_*(X) \subseteq H_1(X)$. We also have the obvious inclusion $(p_1)_* H_*(X) \subseteq p_* H_*(X)$ from which it follows that $(p_1)_* H_*(X) \subseteq p_* H_1(X)$. Suppose that

this last inclusion is not an equality. Then, the idempotent $p - p_1$ acts non-trivially on $H_1(X)$. But we know that $(p_1)_* \text{Alb}_X = p_* \text{Alb}_X$. Applying $(p - p_1)_*$ to this equality gives $(p - p_1)_* \text{Alb}_X = 0$. Therefore $p - p_1$ is a correspondence which acts trivially on Alb_X but non-trivially on $H_1(X)$. This is impossible. \square

Theorem 3.6. *Let $M = (X, p)$ be an effective Chow motive. Then, there exists an idempotent $p_{2d-1} \in p \circ CH_d(X \times X) \circ p$ such that $(p_{2d-1})_* H_*(X) = p_* H_{2d-1}(X)$ and such that $(X, p_{2d-1}) \simeq \mathfrak{h}_1(p_* \text{Pic}_X^0)(d-1)$. The idempotent p_{2d-1} is called the Picard projector.*

Proof. Let q be the Albanese projector for ${}^t p$. Then, $p_{2d-1} := {}^t q$ is the required Picard projector. \square

Remark 3.7. The Albanese projector and the Picard projector are certainly not unique, cf. the proof of theorem 3.9.

Lemma 3.8. *Let $\gamma \in CH^0(V \times W)$ be a correspondence such that γ_* acts trivially on zero-cycles. Then, $\gamma = 0$.*

Proof. We can assume that V and W are both connected. The cycle γ is equal to $a \cdot [V \times W]$ for some $a \in \mathbf{Q}$. Let z be a zero-cycle on V . Then, $\gamma_* z = a \cdot \deg z \cdot [W]$. This immediately implies $a = 0$. \square

Theorem 3.9. *The projectors p_0, p_{2d} , the Albanese projector p_1 and the Picard projector p_{2d-1} can be chosen to be orthogonal.*

Proof. As done in [19, 4.4], it can be immediately checked that the projectors constructed above with respect to the positive zero-cycle $y = \iota_* z$ satisfy $p_0 \circ p_1 = p_1 \circ p_0 = p_0 \circ p_{2d-1} = p_0 \circ p_{2d} = 0$. Therefore, after transposing these equalities, the only possible missing orthogonality relations are between p_1 and p_{2d-1} .

The correspondence p_1 factors through a curve C and the correspondence p_{2d-1} factors through a curve C' . We see that $p_1 \circ p_{2d-1}$ factors through a correspondence $\gamma \in CH_d(C' \times C)$ and that $p_{2d-1} \circ p_1$ factors through a correspondence $\gamma' \in CH_{2-d}(C \times C')$. Therefore, the projectors constructed above are automatically orthogonal if $d > 2$. When $d = 2$, we have $p_1 \circ p_3 = 0$. Indeed, $p_1 \circ p_3$ is a correspondence in $CH_2(C' \times C)$. Moreover, p_3 maps zero-cycles to homologically trivial zero-cycles. Therefore $p_1 \circ p_3$ maps zero-cycles to homologically trivial one-cycles in C , and hence $p_1 \circ p_3$ acts trivially on $CH_0(C')$. It follows from lemma 3.8 that $p_1 \circ p_3 = 0$.

As done by Scholl [19, 4.3] (see also the non-commutative Gram-Schmidt process below), if we set $p'_1 = (1 - \frac{1}{2}p_3) \circ p_1$ and $p'_3 = p_3 \circ (1 - \frac{1}{2}p_1)$, we get a set $\{p_0, p'_1, p'_3, p_4\}$ of mutually orthogonal projectors. In order to conclude, it remains to observe that $p'_1 \circ p_1 \in \text{Hom}((X, p_1), (X, p'_1))$ and $p'_3 \circ p_3 \in \text{Hom}((X, p_3), (X, p'_3))$ are isomorphisms of Chow motives with respective inverses $p_1 \circ p'_1$ and $p_3 \circ p'_3$. \square

Remark 3.10. Had we not constructed p_0 and p_1 with the same choice of a positive degree zero-cycle $\iota_* z$ on X , it would have no longer been true that $p_1 \circ p_0 = 0$. However, we would still have had the relation $p_0 \circ p_1 = 0$. Likewise, the relation $p_{2d-1} \circ p_{2d} = 0$ would hold anyway. The non-commutative Gram-Schmidt process of theorem 4.3 would have then made it possible to orthonormalise the system $\{p_0, p_1, p_{2d-1}, p_{2d}\}$.

As an immediate corollary, we can extend Murre's theorem on surfaces [16] to direct summands of Chow motives of surfaces.

Theorem 3.11. *Let $M = (S, p)$ be a Chow motive where S is a smooth projective surface. Then, M has a Murre decomposition.*

Proof. The correspondences p_0, p_1, p_3 and p_4 of theorem 3.9 together with $p_2 := p - \sum_{i \neq 2} p_i$ give a Chow-Künneth decomposition for M . The fact that such a decomposition satisfies Murre's decomposition is contained in [20, theorem 4.15] together with the fact that clearly, for dimension reasons, p_3 and p_4 act trivially on $CH_0(M)$. \square

4 Self-dual Chow-Künneth decompositions

Let X be a smooth projective variety of dimension d over k . It is proved in [21, theorem 4.2] that if the cohomology of X in degree $\neq d$ is generated by the cohomology of curves, then X admits a self-dual Chow-Künneth decomposition. Precisely if $H_i(X) = N^{\lfloor i/2 \rfloor} H_i(X)$ for all $i \neq d$, where N is the coniveau filtration, then X has a Chow-Künneth decomposition. It follows from theorem 1.3 together with a decomposition of the diagonal argument à la Bloch-Srinivas that a fourfold which is fibred in rationally connected threefolds over a curve has a self-dual Chow-Künneth decomposition [21, corollary 4.7]. Del Angel and Müller-Stach [4] proved that unirational threefolds have a Chow-Künneth decomposition. To do so, they use Mori theory to reduce to the case of a conic fibration. In this section, we generalise their result by proving the following :

Theorem 4.1. *Let $f : X \rightarrow S$ be a dominant morphism defined over a field k from a smooth projective variety X to a smooth projective surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Suppose that X has dimension $d \leq 4$. Then, X has a self-dual Chow-Künneth decomposition $\{\pi_i\}_{0 \leq i \leq 2d}$.*

Moreover, this decomposition can be chosen so as to satisfy the following properties :

- π_0 factors through a point P , i.e. (X, π_0) is isomorphic to $\mathfrak{h}(P)$.
- π_1 and π_3 factor through a curve, i.e. there is a curve C_0 (resp. C_1) such that (X, π_1) (resp. (X, π_3)) is a direct summand of $\mathfrak{h}_1(C_0)$ (resp. $\mathfrak{h}_1(C_1)(1)$).
- π_2 factors through a surface, i.e. there is a surface S' such that (X, π_2) is isomorphic to a direct summand of $\mathfrak{h}(S')$.
- If $d = 4$, π_4 has a representative supported on $X \times D$ for some divisor $D \subset X$.

In particular, this gives an alternate proof to del Angel and Müller-Stach's result for conic fibrations over a surface.

We divide the proof into several steps.

4.1 The projectors π_0 , π_1 and π_2^{tr}

The surface S has a Chow-Künneth decomposition [16, 19] $\{\pi_0^S, \pi_1^S, \pi_2^S, {}^t\pi_1^S, {}^t\pi_0^S\}$. The motive (S, π_2^S) admits a direct summand $(S, \pi_2^{tr,S})$ called its transcendental part, cf [11]. The action of the idempotent $\pi_2^{tr,S}$ on the homology of S is the orthogonal projector on the orthogonal complement for cup-product of the span of the classes of algebraic one-cycles inside $H_2(X)$. In characteristic zero, for Betti cohomology, $(\pi_2^{tr,S})_*H_*(X)$ is the sub-Hodge structure of $H_2(X)$ generated by $H^{2,0}(X) = H^2(X, O_X)$ thanks to the Lefschetz (1,1)-theorem. The idempotent $\pi_2^{tr,S}$ acts trivially on $CH_1(S_\Omega)$ and on $CH_2(S_\Omega)$ so that $CH_*(S, \pi_2^{tr,S}) = CH_0(S, \pi_2^{tr,S})$.

By proposition 1.1, there is a correspondence $\Gamma \in CH^{\dim X}(S \times X)$ such that $\Gamma_f \circ \Gamma = \Delta_S$, so that the Chow motive of S is a direct summand of the Chow motive of X . We thus define mutually orthogonal idempotents $\pi_0 := \Gamma \circ \pi_0^S \circ \Gamma_f$, $\pi_1 := \Gamma \circ \pi_1^S \circ \Gamma_f$ and $\pi_2^{tr} := \Gamma \circ \pi_2^{tr,S} \circ \Gamma_f$. Because the idempotents π_0^S , π_1^S and $\pi_2^{tr,S}$ in $CH_2(S \times S)$ satisfy $(\pi_0^S)_*H_*(S) = H_0(S)$, $(\pi_1^S)_*H_*(S) = H_1(S)$ and $(\pi_2^{tr,S})_*H_*(S) \subset H_2(S)$, we see that $(\pi_0)_*H_*(X) \subset H_0(X)$, $(\pi_1)_*H_*(X) \subset H_1(X)$ and $(\pi_2^{tr})_*H_*(X) \subset H_2(X)$.

Since $CH_0(S_\Omega) = (\pi_0^S + \pi_1^S + \pi_2^{tr,S})_*CH_0(S_\Omega)$ and since both $(\Gamma_f)_* : CH_0(X_\Omega) \rightarrow CH_0(S_\Omega)$ and $\Gamma_* : CH_0(S_\Omega) \rightarrow CH_0(X_\Omega)$ are isomorphisms by theorem 1.3, we see that the decomposition $\mathfrak{h}(X) = (X, \pi_0) \oplus (X, \pi_1) \oplus (X, \pi_2^{tr}) \oplus M$ satisfies $CH_0(M_\Omega) = 0$. Theorem 2.1 gives a smooth projective variety Y of dimension one less than the dimension d of X together with an idempotent $q \in CH_{d-1}(Y \times Y)$ such that $M \simeq (Y, q, 1)$.

By definition $H_*(Y, q, 1) = H_{*-2}(Y, q) = q_*H_{*-2}(Y)$. Consequently we see that $H_0(M) = 0$ and also that $H_1(M) = 0$. Therefore $(\pi_0)_*H_*(X) = H_0(X)$ and $(\pi_1)_*H_*(X) = H_1(X)$.

It is interesting to note that we can show that the π_i 's act as the Künneth projectors on homology only after having determined their action on Chow groups.

4.2 Chow-Künneth decomposition for $\dim X = 3$

When $\dim X = 3$, the dimension of Y is two. Theorem 3.11 then says that M has a Chow-Künneth decomposition. Hence, X has a Chow-Künneth decomposition. This will be made more precise in §4.6.

4.3 The projectors π_2^{alg} and π_3

From now on, unless otherwise stated, the dimension d of X is supposed to be ≥ 4 .

Let $p := \Delta_X - (\pi_0 + \pi_1 + \pi_2^{tr})$. We have the decomposition $\mathfrak{h}(X) = (X, \pi_0) \oplus (X, \pi_1) \oplus (X, \pi_2^{tr}) \oplus M$ with $M = (X, p)$ isomorphic to $(Y, q, 1)$. Choose an isomorphism $f : (Y, q, 1) \rightarrow M$ and let $g : M \rightarrow (Y, q, 1)$ be its inverse. Let π_0^Y and π_1^Y be respectively the point projector and the Albanese projector for $(Y, q, 0)$ (theorem 3.9). We define idempotents $\pi_2^{alg} := f \circ \pi_0^Y \circ g$ and $\pi_3 := f \circ \pi_1^Y \circ g$.

These two idempotents are orthogonal and are obviously orthogonal to the idempotents π_0 , π_1 and π_2^{tr} previously defined. Their action on cohomology is the expected one : we have $H_2(X) = H_2(X, \pi_2^{tr}) \oplus H_2(M)$ but $H_2(M) = H_0(Y, q) = H_0(Y, \pi_0^Y)$. Therefore $\pi_2 := \pi_2^{tr} + \pi_2^{alg}$ induces the Künneth projector $H_*(X) \rightarrow H_2(X) \rightarrow H_*(X)$. We also have $H_3(X) = H_3(M) = H_1(Y, q) = H_1(Y, \pi_1^Y)$ and hence $(\pi_3)_* H_*(X) = H_3(X)$.

4.4 The remaining projectors

We now define $\pi_{2d} := {}^t\pi_0$, $\pi_{2d-1} := {}^t\pi_1$, $\pi_{2d-2} := {}^t\pi_2$ and $\pi_{2d-3} := {}^t\pi_3$. By Poincaré duality, these idempotents satisfy $(\pi_i)_* H_*(X) = H_i(X)$.

4.5 Orthonormalising the projectors

We have the following non-commutative Gram-Schmidt process [21, lemma 2.12]

Lemma 4.2. *Let V be a \mathbf{Q} -algebra and let k be a positive integer. Let π_0, \dots, π_n be idempotents in V such that $\pi_i \circ \pi_j = 0$ whenever $i - j < k$ and $i \neq j$. Then the endomorphisms*

$$p_i := (1 - \frac{1}{2}\pi_n) \circ \dots \circ (1 - \frac{1}{2}\pi_{i+1}) \circ \pi_i \circ (1 - \frac{1}{2}\pi_{i-1}) \circ \dots \circ (1 - \frac{1}{2}\pi_0)$$

define idempotents such that $p_i \circ p_j = 0$ whenever $i - j < k + 1$ and $i \neq j$.

Let's state an orthonormalisation result in our particular case of interest.

Theorem 4.3. *Let X be a smooth projective variety of dimension d . Let $i < d$ be an integer and let $\pi_0, \dots, \pi_i \in CH_d(X \times X)$ be idempotents such that $(\pi_j)_* H_*(X) = H_j(X)$ for all $0 \leq j \leq i$. Let $\pi_{2d-j} := {}^t\pi_j$ for $0 \leq j \leq i$. If $\pi_r \circ \pi_s = 0$ for all $0 \leq r < s \leq 2d$, then the non-commutative Gram-Schmidt process of lemma 4.2 gives mutually orthogonal idempotents $\{p_j\}_{j \in \{0, \dots, i, 2d-i, \dots, 2d\}}$ such that $(p_j)_* H_*(X) = H_j(X)$ and $p_{2d-j} := {}^t p_j$ for all $j \in \{0, \dots, i, 2d-i, \dots, 2d\}$. Moreover, we have isomorphisms of Chow motives $(X, \pi_j) \simeq (X, p_j)$ for all j .*

Proof. In order to get mutually orthogonal idempotents, it is enough to apply lemma 4.2 $2i + 2$ times. In order to prove the theorem, it suffices to prove each statement after each application of the process of lemma 4.2. Everything is then clear, except perhaps for the last statement. The isomorphism is simply given by the correspondence $p_j \circ \pi_j$; its inverse is $\pi_j \circ p_j$ as can be readily checked. \square

We wish to apply theorem 4.3 to the set of idempotents $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_{2d-3}, \pi_{2d-2}, \pi_{2d-1}, \pi_{2d}\}$. In order to do so, we have to show that $\pi_i \circ \pi_j = 0$ whenever $i < j$. We already know that π_0, π_1, π_2 and π_3 are mutually orthogonal. Let's prove the missing orthogonality relations. First we have :

- $\pi_0 \circ {}^t\pi_0 = \pi_0 \circ {}^t\pi_1 = \pi_0 \circ {}^t\pi_2 = \pi_0 \circ {}^t\pi_3 = 0.$
- $\pi_1 \circ {}^t\pi_1 = \pi_1 \circ {}^t\pi_2 = \pi_1 \circ {}^t\pi_3 = 0.$
- $\pi_2 \circ {}^t\pi_2^{alg} = 0.$

These relations are obvious : one uses a dimension argument as well as the fact that π_0 (resp. $\pi_1, \pi_2^{tr}, \pi_2^{alg}, \pi_3$) factors through a variety P (resp. C, S, P', C') of dimension 0 (resp. 1, 2, 0, 1). For instance, $\pi_1 \circ {}^t\pi_3$ factors through a correspondence in $CH_{d-1}(C' \times C)$. If $d \geq 4$, then this last group is trivial.

Using the same arguments, the following orthogonality relations can be further proved. These relations are not necessary to run the non-commutative Gram-Schmidt process.

- ${}^t\pi_0 \circ \pi_0 = {}^t\pi_0 \circ \pi_1 = {}^t\pi_0 \circ \pi_2 = {}^t\pi_0 \circ \pi_3 = 0.$
- ${}^t\pi_1 \circ \pi_1 = {}^t\pi_1 \circ \pi_2 = {}^t\pi_1 \circ \pi_3 = 0.$
- ${}^t\pi_2 \circ \pi_2^{alg} = 0.$
- ${}^t\pi_3 \circ \pi_2^{alg} = 0.$

Secondly, the remaining orthogonality relations needed to run the non-commutative Gram-Schmidt process follow from lemma 3.8.

- $\pi_2 \circ {}^t\pi_2^{tr} = 0.$ The correspondence $\pi_2 \circ {}^t\pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr, S} \in CH_d(S \times S)$. If $d > 4$, then the statement is clear. If $d = 4$, we use the fact that $\pi_2^{tr, S}$ sends zero-cycles on S to zero-cycles in the Albanese kernel of S . Hence $\gamma \circ \pi_2^{tr, S}$ sends zero-cycles on S to homologically trivial two-cycles on S . In particular $\gamma \circ \pi_2^{tr, S}$ acts trivially on zero-cycles on S and we can therefore apply lemma 3.8.
- $\pi_2 \circ {}^t\pi_3 = 0.$ The correspondence $\pi_2 \circ {}^t\pi_3$ factors through a correspondence $\gamma \in CH_{d-1}(C' \times S)$ that sends zero-cycles to homologically trivial cycles on S . Again, if $d > 4$ the result is trivial. If $d = 4$, we conclude by lemma 3.8.
- $\pi_3 \circ {}^t\pi_3 = 0.$ The correspondence $\pi_3 \circ {}^t\pi_3$ factors through a correspondence $\gamma \in CH_{d-2}(C' \times C')$ that sends zero-cycles to homologically trivial cycles on C . Again, if $d > 4$ the result is trivial. If $d = 4$, we conclude by lemma 3.8.

We are now in a position to apply theorem 4.3 to obtain a set of mutually orthogonal idempotents $\{p_0, p_1, p_2, p_3, p_{2d-3}, p_{2d-2}, p_{2d-1}, p_{2d}\}$ such that $p_{2d-i} = {}^t p_i$ which induce the expected Künneth projectors modulo homological equivalence.

Remark 4.4. It follows from the above discussion that the only possible missing orthogonality relations among the idempotents $\pi_0, \pi_1, \pi_2^{alg}, \pi_2^{tr}, \pi_3$ and their transpose are the following.

- ${}^t\pi_3 \circ \pi_2^{tr}$.
- ${}^t\pi_2^{tr} \circ \pi_2^{tr}$.
- ${}^t\pi_3 \circ \pi_3$.

It can then be checked that it is actually enough to run the non-commutative Gram-Schmidt process only once on the set of idempotents $\{\pi_0, \pi_1, \pi_2, \pi_3, \pi_{2d-3}, \pi_{2d-2}, \pi_{2d-1}, \pi_{2d}\}$ to get a set of mutually orthogonal idempotents. We can therefore describe the p_i 's in terms of the π_i 's by not too complicated explicit formulas. Such formulas may then be used for instance to give a quicker proof of the motivic Lefschetz conjecture for X . However, we describe a method that might be useful in other situations where the Gram-Schmidt process needs to be run several times.

The following proposition is fundamental to proving proposition 4.6 and hence to proving Murre's conjectures for X .

Proposition 4.5. *Let p and q be any two distinct idempotents among the idempotents $\pi_0, \pi_1, \pi_2, \pi_3, \pi_{2d-3}, \pi_{2d-2}, \pi_{2d-1}$ and π_{2d} . Then $p \circ q$ acts trivially on $CH_l(X_\Omega)$ for all l .*

Proof. From remark 4.4 we only need to prove that ${}^t\pi_3 \circ \pi_2^{tr}$, ${}^t\pi_2^{tr} \circ \pi_2^{tr}$ and ${}^t\pi_3 \circ \pi_3$ act trivially on $CH_*(X_\Omega)$. In the first case, ${}^t\pi_3 \circ \pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr, S} \in CH_0(S \times C)$ for some curve C and it therefore acts trivially on $CH_*(X_\Omega)$ because $\pi_2^{tr, S}$ only acts non-trivially on $CH_0(S_\Omega)$. In the second case, ${}^t\pi_2^{tr} \circ \pi_2^{tr}$ factors through a correspondence $\gamma \circ \pi_2^{tr, S} \in CH_0(S \times S)$ and we conclude in the same way. In the last case, ${}^t\pi_3 \circ \pi_3$ factors through a correspondence $\gamma \circ \pi_1^C \in CH_0(C \times C)$ which also acts trivially on $CH_*(C_\Omega)$ because π_1^C acts trivially on $CH_1(C_\Omega)$. \square

Proposition 4.6. $(p_0 + p_1 + p_2)_*CH_0(X_\Omega) = CH_0(X_\Omega)$.

Proof. We know that $CH_0(X_\Omega) = (\pi_0 + \pi_1 + \pi_2^{tr})_*CH_0(X_\Omega)$. We also know from theorem 4.3 that there are isomorphisms of Chow motive $(X, \pi_0) \simeq (X, p_0)$ and $(X, \pi_1) \simeq (X, p_1)$, $(X, \pi_2^{tr}) \simeq (X, p_2^{tr})$; it is thus very tempting to conclude that $(p_0 + p_1 + p_2^{tr})_*CH_0(X_\Omega) = (\pi_0 + \pi_1 + \pi_2^{tr})_*CH_0(X_\Omega)$. However this appears not to be obvious at all and a careful analysis of the non-commutative Gram-Schmidt process needs to be carried on. By examining the formula defining the idempotents p_i for $0 \leq i \leq 2$, we see thanks to proposition 4.5 that orthonormalising the family of idempotents does not alter their action on the Chow groups of X , i.e. for $x \in CH_*(X_\Omega)$ we have $(p_i)_*x = (\pi_i)_*x \in CH_l(X_\Omega)$. This yields $(p_0 + p_1 + p_2^{tr})_*CH_0(X_\Omega) = CH_0(X_\Omega)$ as claimed. \square

Finally, when $d = 4$, we define $p_4 := \Delta_X - \sum_{i \neq 4} p_i$. The set $\{p_i\}_{i \neq 4}$ is then a self-dual Chow-Künneth decomposition for X . Moreover, p_4 has the following property.

Proposition 4.7. *The idempotent p_4 is supported on $X \times D$ for some divisor $D \subset X$.*

Proof. Proposition 4.6 implies that $(p_4)_* CH_0(X_\Omega) = 0$. Therefore, as in the proof of theorem 2.1, we get that p_4 is supported on $D \times X$ for some divisor D of X . Because ${}^t p_d = p_d$, we see that p_d has a representative supported on $X \times D$. \square

4.6 Back to the case $\dim X = 3$

Let's now consider the case of a conic fibration over a surface. In section 4.2, we already gave a quick argument showing that X has a Chow-Künneth decomposition. We would like to make this more precise. First we want to show that a Chow-Künneth decomposition for X can be chosen to be self-dual, a result which is not shown in [4]. Secondly, in order to prove Murre's conjectures for such a decomposition (which will be done in section 6) we want to show that p_3 factors through a curve.

For this purpose, we define $\pi_0, \pi_1, \pi_2^{tr}, \pi_2^{alg}$ and $\pi_2 := \pi_2^{tr} + \pi_2^{alg}$ the same way we did in sections 4.1 and 4.3. We then define $\pi_6 = {}^t \pi_0, \pi_5 = {}^t \pi_1$ and $\pi_4 = {}^t \pi_2$. As before, it is easy to see that these do define the Künneth projectors in homology.

In order to exhibit a self-dual Chow-Künneth decomposition for X , we need to show that $\pi_i \circ \pi_j = 0$ for all $i < j$ not equal to 3. The only case not covered by the methods of section 4.5 is $\pi_2^{tr} \circ {}^t \pi_2^{tr} = 0$. There are two ways of proving this. The first way is particular to our present situation while the second is adaptable to more general situations and will be needed in order to prove the motivic Lefschetz conjecture for X in §5.

On the one hand we have

Lemma 4.8. *Let $f : X \rightarrow S$ be a dominant map between two smooth projective varieties with $\dim X > \dim S$. Then $\Gamma_f \circ {}^t \Gamma_f = 0$.*

Proof. By definition we have $\Gamma_f \circ {}^t \Gamma_f = (p_{1,3})_*(p_{1,2}^* {}^t \Gamma_f \cap p_{2,3}^* \Gamma_f)$, where $p_{i,j}$ denotes projection from $S \times X \times S$ to the (i,j) -th factor. These projections are flat morphisms, therefore by flat pullback we have $p_{1,2}^* {}^t \Gamma_f = [{}^t \Gamma_f \times S]$ and $p_{2,3}^* \Gamma_f = [S \times \Gamma_f]$. It is easy to see that the closed subschemes ${}^t \Gamma_f \times S$ and $S \times \Gamma_f$ of $S \times X \times S$ intersect properly. Their intersection is given by $\{(f(x), x, f(x)) : x \in X\} \subset S \times X \times S$. This is a closed subset of dimension $\dim X$ and its image under the projection $p_{1,3}$ has dimension $\dim S$, which is strictly less than $\dim X$ by assumption. The projection $p_{1,3}$ is a proper map and hence by proper pushforward we get that $(p_{1,3})_*[\{(f(x), x, f(x)) \in S \times X \times S : x \in X\}] = 0$. \square

On the other hand we have the following analogue of lemma 3.8.

Lemma 4.9. *Let $\gamma \in CH^1(V \times W)$ be a correspondence such that both γ_* and γ^* act trivially on zero-cycles. Then, $\gamma = 0$.*

Proof. We can assume V and W are connected. We have $\text{Pic}(V \times W) = \text{Pic}(V) \times [W] \oplus [V] \times \text{Pic}(W)$. The cycle γ is thus equal to $D_1 \times [W] \oplus [V] \times D_2$ for some divisors $D_1 \in CH^1(V)$ and $D_2 \in CH^1(W)$. Let z be a zero-cycle on V . Then, $\gamma_* z = \deg z \cdot D_2$. This immediately implies $D_2 = 0$. Likewise, if $z \in CH_0(W)$, $\gamma^* z = 0$ implies $D_1 = 0$. We have thus proved that $\gamma = 0$. \square

Proposition 4.10. $\pi_2^{tr} \circ {}^t\pi_2^{tr} = 0$.

Proof. From lemma 4.8 and from the very definition of π_2^{tr} the result is immediate. Let's now give a proof using lemma 4.9 when the base field has characteristic zero. The reason is that we use Abel-Jacobi maps (although it is almost certainly true that the Albanese variety and the Picard variety enjoy the required functoriality properties over any base field). The correspondence $\pi_2^{tr} \circ {}^t\pi_2^{tr} \in CH_3(X \times X)$ factors through a correspondence $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S} \in CH^1(S \times S)$. In particular, by functoriality of the Abel-Jacobi map, $\pi_2^{tr} \circ {}^t\pi_2^{tr}$ sends 0-cycles on S to 1-cycles on S in the kernel of the Abel-Jacobi map. This last kernel is trivial. Therefore $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S}$ acts trivially on zero-cycles. Clearly the same holds for its transpose. Therefore $\pi_2^{tr,S} \circ \gamma \circ {}^t\pi_2^{tr,S} = 0$ and hence $\pi_2^{tr} \circ {}^t\pi_2^{tr} = 0$. \square

Running theorem 4.3, we get mutually orthogonal idempotents $p_0, p_1, p_2, p_4 = {}^t p_2, p_5 = {}^t p_1$ and $p_6 = {}^t p_0$. We then have a similar statement as proposition 4.6.

Proposition 4.11. $(p_0 + p_1 + p_2)_* CH_0(X_\Omega) = CH_0(X_\Omega)$.

Proof. The proof uses the same arguments as the ones appearing in the proof of proposition 4.6. It consists in proving that ${}^t q \circ p$ acts trivially on $CH_*(X_\Omega)$ for any p and q among the idempotents $\pi_0, \pi_1, \pi_2^{alg}$ and π_2^{tr} . \square

Setting $p_3 := \Delta_X - \sum_{i \neq 3} p_i$, we get a self-dual Chow-Künneth decomposition for X . The middle idempotent p_3 has the following property.

Proposition 4.12. *The idempotent p_3 factors through a curve. Precisely, there exists a curve C such that (X, p_3) is isomorphic to a direct summand of $\mathfrak{h}_1(C)(1)$.*

Proof. Proposition 4.11 shows that $CH_0(X_\Omega, p_3) = 0$. Then, a similar argument to the one appearing in the proof of theorem 2.1 shows that p_3 is supported on $D \times X$ for some divisor D . Because $p_3 = {}^t p_3$, p_3 has a representative actually supported on $X \times D$. We thus see, after altering D that the action of p_3 on $CH_1(X_\Omega)$ factors through the CH_1 of a surface. This last group is representable. Moreover, it is easy to see that p_3 acts trivially on $CH_2(X_\Omega)$ and on $CH_3(X_\Omega)$. Therefore $CH_*(X, p_3)_{alg}$ is equal to $CH_1(X, p_3)_{alg}$ and is representable. The proposition then follows from [22, Th. 3.4]. \square

5 The motivic Lefschetz conjecture for X

Let X be a smooth projective variety of dimension d over a field k . Let $i \leq d$ and let $\iota : H \rightarrow X$ be a smooth linear section of dimension i and let $L := (\iota, \text{id}_X)_* \Gamma_\iota = \Gamma_\iota \circ {}^t \Gamma_\iota \in CH_i(X \times X)$. The correspondence L acts on cohomology or Chow groups as intersecting $d - i$ times by a smooth hyperplane section of X . The variety X is said to satisfy the motivic Lefschetz conjecture in degree i if there exist mutually orthogonal idempotents π_i and π_{2d-i} such that $H_*(X, \pi_i) = H_i(X)$ and $H_*(X, \pi_{2d-i}) = H_{2d-i}(X)$ and such that the induced map

$$L : (X, \pi_{2d-i}) \rightarrow (X, \pi_i, d - i)$$

is an isomorphism of Chow motives. The variety X is said to satisfy the motivic Lefschetz conjecture if it satisfies the motivic Lefschetz conjecture in all degrees $< d$. Note that if X satisfies the motivic Lefschetz conjecture in degree i then X satisfies the Lefschetz standard conjecture in degree i , i.e. there exists a correspondence $\Gamma \in CH^i(X \times X)$ such that $\Gamma_* : H_i(X) \rightarrow H_{2d-i}(X)$ is the inverse to $L : H_{2d-i}(X) \rightarrow H_i(X)$. The motivic Lefschetz conjecture for X follows from a combination of the Lefschetz standard conjecture for X and of Kimura's finite dimensionality conjecture for X ; it is thus expected to hold for all smooth projective varieties.

Proposition 5.1. *Let P be a zero-dimensional variety over k . Let $p \in CH_d(X \times X)$ be an idempotent such that (X, p) is isomorphic to $\mathfrak{h}(P)(i)$ for some integer i satisfying $2i \leq d$. If the induced map $L : H_{2d-2i}(X, {}^t p) \rightarrow H_{2i}(X, p)$ is an isomorphism, then $L : (X, {}^t p) \rightarrow (X, p, d - 2i)$ is an isomorphism of Chow motives.*

Proof. There exist correspondences $f \in \text{Hom}(\mathfrak{h}(P)(i), (X, p))$ and $g \in \text{Hom}((X, p), \mathfrak{h}(P)(i))$ such that $g \circ f = \text{id}_{\mathfrak{h}(P)(i)}$ and $f \circ g = p$. The correspondence $g \circ L \circ {}^t g \in \text{End}(\mathfrak{h}(P))$ induces an automorphism of $H_0(P)$ and hence is itself an automorphism. Therefore, it admits an inverse $\alpha \in \text{End}(\mathfrak{h}(P))$. It is now straightforward to check that ${}^t p \circ {}^t g \circ \alpha \circ g \circ p$ is the inverse of $p \circ L \circ {}^t p$. \square

Proposition 5.2. *Let J be an abelian variety over k . Let $p \in CH_d(X \times X)$ be an idempotent such that (X, p) is isomorphic to $\mathfrak{h}_1(J)(i)$ for some integer i satisfying $2i + 1 \leq d$ and such that p is orthogonal to ${}^t p$ (this last condition is automatically satisfied if $2i + 1 < d - 1$). If the induced map $L : H_{2d-2i-1}(X, {}^t p) \rightarrow H_{2i+1}(X, p)$ is an isomorphism, then $L : (X, {}^t p) \rightarrow (X, p, d - 2i - 1)$ is an isomorphism of Chow motives.*

Proof. There exist correspondences $f \in \text{Hom}(\mathfrak{h}_1(J)(i), (X, p))$ and $g \in \text{Hom}((X, p), \mathfrak{h}_1(J)(i))$ such that $g \circ f = \text{id}_{\mathfrak{h}_1(J)(i)}$ and $f \circ g = p$. The correspondence $g \circ L \circ {}^t g \in \text{End}(\mathfrak{h}_1(J))$ induces an automorphism of $H_1(J)$ and hence is itself an automorphism (indeed by [19, Prop. 4.5] we have $\text{End}(\mathfrak{h}_1(J)) = \text{End}_k(J) \otimes \mathbf{Q}$ and it is well-known that a map between abelian varieties which induces an isomorphism in degree one homology must be an isogeny). Therefore, it admits an inverse $\alpha \in \text{End}(\mathfrak{h}_1(J))$. It is now straightforward to check that ${}^t p \circ {}^t g \circ \alpha \circ g \circ p$ is the inverse of $p \circ L \circ {}^t p$. \square

As already proved by Scholl [19], every smooth projective variety satisfies the motivic Lefschetz conjecture in degrees ≤ 1 .

Theorem 5.3. *Let $f : X \rightarrow S$ be a dominant morphism defined over a field k from a smooth projective variety X to a smooth projective surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Then, X satisfies the motivic Lefschetz conjecture in degrees ≤ 3 . In particular, if X has dimension ≤ 4 , then X satisfies the motivic Lefschetz conjecture and hence the Lefschetz standard conjecture.*

Proof. By the construction given in §4, p_0 factors through a point, and p_1 and p_3 factor through the \mathfrak{h}_1 of a curve. The hard Lefschetz theorem says that the map $H_{2d-i}(X) \rightarrow H_i(X)$ induced by intersecting $d-i$ times with a smooth hyperplane section is an isomorphism. Therefore, the two propositions above give the motivic Lefschetz conjecture in degrees 0, 1 and 3 for X .

Let π_2^{tr} be the idempotent of section 4.1. Let's prove that $L : (X, {}^t\pi_2^{tr}, 0) \rightarrow (X, \pi_2^{tr}, d-2)$ is an isomorphism of Chow motives. Because $\iota : H \rightarrow X$ is a linear section of X of dimension 2, lemma 1.1 gives a non-zero integer m such that $\Gamma_f \circ L \circ {}^t\Gamma_f = m \cdot \Delta_S$. It is then straightforward to check that $\frac{1}{m} \cdot {}^t\pi_2^{tr} \circ {}^t\Gamma_f \circ \Gamma_f \circ \pi_2^{tr}$ is the inverse of $\pi_2^{tr} \circ L \circ {}^t\pi_2^{tr}$.

Let π_2^{alg} be the idempotent of section 4.3. Because $L : (X, {}^t\pi_2^{tr}, 0) \rightarrow (X, \pi_2^{tr}, d-2)$ is an isomorphism and because $L_* : H_{2d-2}(X) \rightarrow H_2(X)$ is an isomorphism by the hard Lefschetz theorem, we see that L induces an isomorphism $L : H_{2d-2}(X, {}^t\pi_2^{alg}) \rightarrow H_2(X, \pi_2^{alg})$. Proposition 5.1 then shows that $\pi_2^{alg} \circ L \circ {}^t\pi_2^{alg} \in \text{Hom}((X, {}^t\pi_2^{alg}), (X, \pi_2^{alg}, d-2))$ is an isomorphism.

We have thus showed that $\pi_2 \circ L \circ {}^t\pi_2 \in \text{Hom}((X, {}^t\pi_2), (X, \pi_2, d-2))$ is an isomorphism. Since by theorem 4.3 we know that $(X, p_2, d-2) \simeq (X, \pi_2, d-2)$ and $(X, {}^t p_2) \simeq (X, {}^t\pi_2)$ we get that $(X, p_2, d-2)$ is isomorphic to $(X, {}^t p_2)$. However the isomorphism is induced by $p_2 \circ \pi_2 \circ L \circ {}^t\pi_2 \circ {}^t p_2$ which is not quite the isomorphism we were aiming at.

By remark 4.4, it can be checked that in our particular setting we have $p_2 \circ \pi_2 = p_2$ so that $p_2 \circ L \circ {}^t p_2$ is an isomorphism with inverse $\frac{1}{m} \cdot {}^t p_2 \circ {}^t\pi_2 \circ {}^t\Gamma_f \circ \Gamma_f \circ \pi_2 \circ p_2$.

Let's however give another proof that $p_2 \circ L \circ {}^t p_2$ is an isomorphism that might be useful in other situations. Let q_i be the idempotents obtained from the π_i 's after having applied n times the non-commutative Gram-Schmidt process of lemma 4.2 and let p_i be the idempotents obtained from the q_i 's after one application of the Gram-Schmidt process. we see that $p_2 = q \circ q_2 \circ (1 - \frac{1}{2}q_1) \circ (1 - \frac{1}{2}q_0)$, where q is a correspondence in $CH_d(X \times X)$. Thanks to theorem 4.3, we have isomorphisms $(X, q_i) \simeq (X, \pi_i)$ for $i = 0, 1$ and 2. In particular (X, q_1) is isomorphic to a direct summand of $\mathfrak{h}_1(C)$ for some curve C and (X, q_0) is isomorphic to $\mathfrak{h}(P)$ for some zero-dimensional P . Moreover the isomorphism $(X, q_2) \simeq (X, \pi_2)$ given by theorem 4.3 makes it possible to decompose q_2 as the orthogonal sum $q_2^{alg} + q_2^{tr}$ accordingly with the decomposition $\pi_2 = \pi_2^{alg} + \pi_2^{tr}$. We can conclude that $p_2 \circ L \circ {}^t p_2$ is an isomorphism if we can show that it is equal to $p_2 \circ \pi_2 \circ L \circ {}^t\pi_2 \circ {}^t p_2$. For

this purpose it is enough to check that for all correspondences $\alpha \in CH_2(X \times X)$ we have $q_r \circ \alpha \circ {}^t q_2 = 0$ for $r = 0, 1$. When $r = 0$, $q_0 \circ \alpha \circ {}^t q_2^{alg}$ factors through a correspondence $\gamma \in CH_1(P' \times P)$ for some zero-dimensional P' and is thus zero for dimension reasons and $q_0 \circ \alpha \circ {}^t q_2^{tr}$ factors through a correspondence $\gamma \in CH^0(S \times P)$ with $\gamma^* z = 0$ for any $z \in CH_0(P)$. Lemma 3.8 then shows that $\gamma = 0$ and hence $q_0 \circ \alpha \circ {}^t q_2^{tr} = 0$. When $r = 1$, on the one hand we have that $q_1 \circ \alpha \circ {}^t q_2^{alg}$ factors through a correspondence $\gamma \in CH^0(P' \times C)$ with $\gamma^* z = 0$ for any zero-cycle z on C . Lemma 3.8 then shows that $\gamma = 0$ and hence $q_1 \circ \alpha \circ {}^t q_2^{alg} = 0$. On the other hand, $q_1 \circ \alpha \circ {}^t q_2^{tr}$ factors through a correspondence $\gamma \in CH^1(S \times C)$ with $\gamma^* z = 0$ for any zero-cycle z on C and $\gamma_* z'$ for any zero-cycle z' on S by functoriality of the Abel-Jacobi map. Therefore thanks to lemma 4.9, we get $\gamma = 0$ and hence $q_1 \circ \alpha \circ {}^t q_2^{tr} = 0$. \square

Remark 5.4. The results of this section actually show that for X as in the theorem above and for the idempotents p_i constructed in §4 the map $L : (X, p_{2d-i}) \rightarrow (X, p_i, d-i)$ is an isomorphism for $i \leq 3$ for any choice of a smooth linear section $\iota : H \hookrightarrow X$ of dimension i .

6 Murre's conjectures for X

As shown by Jannsen [9], Murre's conjectures [17] are equivalent to Bloch and Beilinson's. Let's recall them :

- (A) X has a Chow-Künneth decomposition $\{\pi_0, \dots, \pi_{2d}\}$: There exist mutually orthogonal idempotents $\pi_0, \dots, \pi_{2d} \in CH_d(X \times X)$ adding to the identity such that $(\pi_i)_* H_*(X) = H_i(X)$ for all i .
- (B) $\pi_0, \dots, \pi_{2l-1}, \pi_{d+l+1}, \dots, \pi_{2d}$ act trivially on $CH_l(X)$ for all l .
- (C) $F^i CH_l(X) := \text{Ker}(\pi_{2l}) \cap \dots \cap \text{Ker}(\pi_{2l+i-1})$ doesn't depend on the choice of the π_j 's. Here the π_j 's are acting on $CH_l(X)$.
- (D) $F^1 CH_l(X) = CH_l(X)_{\text{hom}}$.

Definition 6.1. A smooth projective variety X of dimension d is said to have a *special* Chow-Künneth decomposition $\{\pi_i\}_{0 \leq i \leq 2d}$ if

- π_{2i} factors through a surface for $2i \neq d$, i.e. there is a surface S_i such that (X, π_{2i}) is a direct summand of $\mathfrak{h}(S_i)(i-1)$.
- π_{2i+1} factors through a curve for $2i+1 \neq d$, i.e. there is a curve C_i such that (X, π_{2i+1}) is a direct summand of $\mathfrak{h}_1(C_i)(i)$.
- π_d has a representative supported on $X \times Z$ as well as a representative supported on $Z' \times X$ for some closed subschemes $Z, Z' \subset X$ of dimension $\lfloor \frac{d}{2} \rfloor + 1$.

The arguments in the proof of the following proposition are essentially contained in [20].

Proposition 6.2. *Let X be a smooth projective variety of dimension $d \leq 4$ that has a special Chow-Künneth decomposition. Then, homological and algebraic equivalence agree on X , X satisfies Murre's conjectures (A), (B) and (D) and the filtration F on $CH_l(X)$ does not depend on the choice of a special Chow-Künneth decomposition for X .*

Proof. Let's first prove that π_{2l} and π_{2l+1} act trivially on $CH_{l+1}(X)$ for all l . Indeed, their action on $CH_{l+1}(X)$ factors through $CH^1(Y)$ for some smooth projective variety Y , may it be C_l , S_l or a resolution of singularities of Z' . By functoriality of the cycle class map to Deligne cohomology, we have a commutative diagram

$$\begin{array}{ccccc} CH_{l+1}(X) & \longrightarrow & CH^1(Y) & \longrightarrow & CH_{l+1}(X) \\ \downarrow & & \downarrow \simeq & & \downarrow \\ H_{\mathcal{D}}^{2d-2l-2}(X, \mathbf{Q}(d-l-1)) & \longrightarrow & H_{\mathcal{D}}^2(Y, \mathbf{Q}(1)) & \longrightarrow & H_{\mathcal{D}}^{2d-2l-2}(X, \mathbf{Q}(d-l-1)). \end{array}$$

The composition of the two bottom horizontal map is zero. This is because π_{2l} and π_{2l+1} act trivially on $H_{2l+2}(X)$ and $H_{2l+3}(X)$ and because $H_{\mathcal{D}}^{2d-2l-2}(X, \mathbf{Q}(d-l-1))$ is an extension of the Hodge classes in $H_{2l+2}(X)$ by the intermediate Jacobian $J_l(X)$ which is a quotient of $H_{2l+3}(X, \mathbf{C})$. An easy diagram chase then shows that π_{2l} and π_{2l+1} act trivially on $CH_{l+1}(X)$.

As such, the idempotent π_{2l+1} acts non-trivially only on $CH_l(X)$ and the idempotent π_{2l} acts non-trivially only on $CH_{l-1}(X)$ and on $CH_l(X)$. Therefore conjecture (B) holds for the Chow-Künneth decomposition given by the π_l 's. We also see that the action of π_l on $CH_i(X)$, if not zero, factors through the CH_0 or the CH^1 of a smooth projective varieties. Since algebraic and homological equivalence agree for zero-cycles and for codimension-one cycles, we get that they agree on $CH_i(X)$ for all i .

By functoriality of the cycle class map with respect to the action of correspondences combined with the fact that π_{2l} acts as the identity on $H_{2l}(X)$ immediately implies that $\text{Ker}(\pi_{2l}) \subseteq CH_l(X)_{\text{hom}}$. By assumption on π_{2l} , we have that π_{2l} is supported on $X \times Y_l$ for some closed subscheme Y_l of X of dimension $l+1$ (except for $l = \dim X$ in which case conjecture (D) for $CH_{\dim X}(X)$ is clear). Let $\tilde{Y}_l \rightarrow Y_l$ be a desingularisation. Then we can write $\pi_{2l} = f \circ g$ for some $f \in \text{Hom}(\mathfrak{h}(\tilde{Y}_l), \mathfrak{h}(X))$ and some $g \in \text{Hom}(\mathfrak{h}(X), \mathfrak{h}(\tilde{Y}_l))$. By functoriality of the Abel-Jacobi map, we have the commutative diagram :

$$\begin{array}{ccccc} CH_l(X)_{\text{hom}} & \xrightarrow{g_*} & CH^1(\tilde{Y}_l)_{\text{hom}} & \xrightarrow{f_*} & CH_l(X)_{\text{hom}} \\ \downarrow AJ_l & & \downarrow \simeq & & \downarrow AJ_l \\ J_l(X)(\mathbf{C}) & \longrightarrow & \text{Pic}_{\tilde{Y}_l}^0(\mathbf{C}) & \longrightarrow & J_l(X)(\mathbf{C}). \end{array}$$

The composition of the two bottom arrows is zero because π_{2l} acts trivially on $H_{2l+1}(X)$. Therefore, if $\alpha \in CH_l(X)_{\text{hom}}$, then we have $(g \circ f \circ g)_* \alpha = 0$ and hence $f_* \circ (g \circ f \circ g)_* \alpha = 0$, i.e. $(\pi_{2l} \circ \pi_{2l})_* \alpha = 0$, that is $(\pi_{2l})_* \alpha = 0$. We have thus established conjecture (D).

Let $\{\pi_i\}_{0 \leq i \leq 2d}$ be a Chow-Künneth decomposition as in the proposition. We have showed that if F is the filtration on $CH_l(X)$ induced by the π_i 's, then $F^1 CH_l(X) = CH_l(X)_{\text{hom}}$ and $F^3 CH_l(X) = 0$. We now show that

$$(1) \quad F^2 CH_l(X) = \text{Ker}(AJ_l : CH_l(X)_{\text{hom}} \rightarrow J_l(X)(\mathbf{C}))$$

and hence that the filtration F does not depend on the π_i 's. Each of the idempotents π_{2l+1} are supported on $X \times Z_l$ for some closed subscheme Z_l of X of dimension $l+1$. Let $\tilde{Z}_l \rightarrow Z_l$ be a desingularisation. Then we can write $\pi_{2l+1} = f \circ g$ for some $f \in \text{Hom}(\mathfrak{h}(\tilde{Z}_l), \mathfrak{h}(X))$ and some $g \in \text{Hom}(\mathfrak{h}(X), \mathfrak{h}(\tilde{Z}_l))$. By functoriality of the Abel-Jacobi map, we have the commutative diagram :

$$\begin{array}{ccccc} CH_l(X)_{\text{hom}} & \xrightarrow{g_*} & CH^1(\tilde{Z}_l)_{\text{hom}} & \xrightarrow{f_*} & CH_l(X)_{\text{hom}} \\ \downarrow AJ_l & & \downarrow \simeq & & \downarrow AJ_l \\ J_l(X)(\mathbf{C}) & \longrightarrow & \text{Pic}_{\tilde{Z}_l}^0(\mathbf{C}) & \longrightarrow & J_l(X)(\mathbf{C}). \end{array}$$

The composition of the two bottom arrows is the identity because π_{2l+1} acts as the identity on $H_{2l+1}(X)$. A diagram chase establishes (1). \square

Proposition 6.3. *Let X be a smooth projective variety of dimension $d \leq 4$ that has a special Chow-Künneth decomposition. If X is Kimura finite-dimensional [12], then X satisfies Murre's conjecture (C).*

Proof. By definition of a special Chow-Künneth decomposition, we see that the cohomology of X is generated through the action of correspondences by the H_1 of curves in odd degrees and by the H_2 of surfaces in even degrees. For example, the idempotent π_4 has a representative supported on $Z \times X$ where Z is some closed subset of X of codimension one. If $\tilde{Z} \rightarrow Z$ is a resolution of singularities of Z , then we see that π_4 factors through a correspondence $f \in CH_4(\tilde{Z} \times X)$, so that $H_4(X) = f_* H_2(\tilde{Z})$. By the Lefschetz hyperplane section theorem, we thus see that $H_4(X)$ is spanned by the H_2 of a surface (namely, a smooth hyperplane section of \tilde{Z}). The variety X is assumed to be Kimura finite-dimensional, therefore we can conclude by [20, Theorem 4.8]. \square

Theorem 6.4. *Let $f : X \rightarrow S$ be a dominant morphism defined over a field k between a smooth projective variety X of dimension ≤ 4 and a smooth projective surface S such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Then X has a special Chow-Künneth decomposition which is self-dual and which satisfies Murre's conjectures (B) and (D). Moreover, the induced filtration F on $CH_l(X)$ does not depend on the choice of a special Chow-Künneth decomposition for X , and if X is Kimura finite-dimensional then F does not depend on the choice of a Chow-Künneth decomposition for X .*

Proof. Theorem 4.1 says that X has a special Chow-Künneth decomposition which is self-dual. We conclude with proposition 6.2 and proposition 6.3. \square

Corollary 6.5. *Let $f : X \rightarrow S$ be a dominant morphism defined over a field k from a smooth projective fourfold X to a smooth projective surface S . Assume that a general fibre of f is (separably) rationally connected. Then X has a self-dual Murre decomposition and X satisfies the motivic Lefschetz conjecture.*

Proof. It is classical that the Chow group of zero-cycles of a rationally connected variety is trivial. \square

7 Murre's conjectures for $X \times C$

Let's consider the following conjecture which is weaker than conjecture (B) of Murre.

(B') The variety X has a Chow-Künneth decomposition $\{p_l : 0 \leq l \leq \dim X\}$ such that p_l acts trivially on $CH_i(X)$ for $l < 2i$.

Let $f : X \rightarrow S$ be a dominant morphism from a smooth projective fourfold to a smooth projective surface such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Consider the self-dual Murre decomposition $\{p_i : 0 \leq i \leq 8\}$ of X given in section 4. Let C be a smooth projective curve and let $\{p_0^C, p_1^C, p_2^C\}$ be a self-dual Chow-Künneth decomposition for C as described in [19]. Then the variety $X \times C$ has a self-dual Chow-Künneth decomposition given by $q_l := \sum_{i+j=l} p_i \times p_j^C$. The results of [20] make it possible to prove the following.

Theorem 7.1. *The fivefold $X \times C$ endowed with the above Chow-Künneth decomposition satisfies Murre's conjectures (A), (B') and (D).*

Proof. The idempotents $p_0, p_1, p_2, p_3, p_0^C$ and p_1^C factor through varieties of respective dimension 0, 1, 2, 1, 0 and 1. As for the idempotent p_4 , it is supported on $X \times D$ for some divisor D in X . It follows that q_l is supported on $(X \times C) \times Z_l$ where Z_l is a closed subscheme of $X \times C$ of dimension at most $\frac{l}{2} + 1$ if l is even and at most $\frac{l-1}{2} + 2$ if l is odd. Thus, $X \times C$ belongs to the set G of [20, 4.3]. It follows from [20, theorem 4.15] that, with respect to the Chow-Künneth decomposition given by the q_l 's, the variety $X \times C$ satisfies Murre's conjectures (B') and (D). \square

Remark 7.2. If we could prove that p_4 factors through a surface, then it would be possible to prove Murre's conjecture (B) for $X \times C$.

8 Application to the finite dimensionality problem

Kimura [12] introduced the notion of finite dimensionality for Chow motives. There he proved that any variety dominated by a product of curves is finite dimensional. It was proved by Guletskii and Pedrini [8] that a surface with representable Chow group of zero-cycles is Kimura finite dimensional. Gorchinskiy and Guletskii [6] then proved that a threefold with representable Chow group of zero-cycle is Kimura finite dimensional. This was subsequently generalised to varieties of any dimension in [21] and to pure motives in [22]. In their paper, Gorchinskiy and Guletskii also prove [6, Theorem 15] that when X is fibred over a curve by Del Pezzo or Enriques surfaces over an algebraically closed field of characteristic zero, then X has representable Chow group of zero-cycles. Their method involves looking at the singular fibres of the family. Our theorem 1.3 is more general and immediately gives

Theorem 8.1. *Let X be a smooth projective threefold over a field k and let $f : X \rightarrow C$ be a dominant morphism over a curve C such that the general fibre of f_Ω has trivial Chow group of zero-cycles. Then X has representable Chow group of zero-cycles and is finite dimensional in the sense of Kimura.*

Godeaux surfaces are examples of surfaces of general type with trivial Chow group of zero-cycles [24]. Therefore new cases encompassed by the above theorem are given by threefolds fibred by Godeaux surfaces over a curve. Let's make this more precise.

Let ζ be a primitive fifth root of unity. The group $G = \mathbf{Z}/5\mathbf{Z}$ acts on \mathbf{P}_k^3 in the following way : $\zeta \cdot [x_0 : x_1 : x_2 : x_3] = [x_0 : \zeta x_1 : \zeta^2 x_2 : \zeta^3 x_3]$. Let $\bar{V} := H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(5))^G$ be the subspace of $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(5))$ consisting of elements invariant under the action of G and let $V \hookrightarrow \bar{V}$ be the Zariski open subset of \bar{V} consisting of elements defining smooth quintic surfaces. If Y_v is a smooth quintic in \mathbf{P}^3 given by the equation $v \in V$, then a local computation shows that Y_v cannot contain the fixed points of the action of G on \mathbf{P}^3 , so that the action of G restricts to a free action on Y_v . The quotient space $X_v := Y_v/G$ is a smooth projective surface called a Godeaux surface. These Godeaux surfaces fit into a family $X \rightarrow \mathbf{P}(\bar{V})$.

Let's consider a smooth projective curve C in $\mathbf{P}(\bar{V})$ that meets $\mathbf{P}(\bar{V} - V)$ transversely. Then, X restricted to C gives a smooth projective threefold $X|_C \rightarrow C$ with general fibre a Godeaux surface. If C is of general type ($g(C) \geq 2$), then by a result of Viehweg [23] which is a special instance of the Iitaka conjecture, $\bar{X}|_C$ is a threefold of general type. We have thus exhibited new examples of threefolds of general type with representable Chow group of zero-cycles (obvious examples are given by the product of a curve of general type with a Godeaux surface). Such threefolds are also Kimura finite dimensional thanks to [6].

In the following theorem, by conic fibration, we mean a dominant morphism $X \rightarrow S$ whose general fibre is a conic.

Theorem 8.2. *Let X be a smooth projective threefold which is a conic fibration over a surface S which is Kimura finite dimensional. Then X is finite dimensional in the sense of Kimura.*

Proof. In section 4, we proved that there is an orthogonal decomposition of the diagonal $\Delta_X = p_0 + p_1 + p_2^{tr} + p_2^{alg} + p_3 + {}^t p_2^{alg} + {}^t p_2^{tr} + {}^t p_1 + {}^t p_0$ with (X, p_0) and (X, p_2^{alg}) isomorphic to twisted motives of points, (X, p_1) and (X, p_3) isomorphic to direct summands of twisted motives of curves; and with (X, p_2^{tr}) isomorphic to $(S, \pi_2^{tr, S})$. Motives of points and motives of curves are finite dimensional [12]. Since S is Kimura finite dimensional by assumption and since finite dimensionality is stable under direct summand [12], we have that (X, p_2^{tr}) is finite dimensional. Therefore X is Kimura finite dimensional. \square

Theorem 8.3. *Let X be as in theorem 8.1 or as in theorem 8.2. Then, X satisfies Murre's conjectures (A), (B), (C) and (D).*

Proof. By theorem 6.4, it remains to prove that X satisfies Murre's conjecture (C). Let $\{p_i, 0 \leq i \leq 6\}$ be a special Chow-Künneth decomposition for X . Such a decomposition exists for X as in theorem 8.1 by [20, Theorem 4.30] together with the fact that a variety with representable Chow group of zero-cycles has vanishing $H^2(X, O_X)$, and it exists for X as in theorem 8.2 by theorem 6.4. The arguments of the proof of proposition 4.12 show that (X, p_3) is isomorphic to the direct summand of the motive of a curve. Moreover, the motive of X is finite-dimensional. The assumptions of [20, Theorem 4.8] are thus met and we can therefore conclude. \square

9 A fourfold of general type satisfying Murre's conjectures

In this section we wish to give explicit examples of fourfolds satisfying the assumptions of theorem 1. For this purpose we consider two-dimensional families of surfaces having trivial Chow group of zero-cycles.

A first type of such families are given by rationally connected fibrations over a surface (separably rationally connected if the base field has positive characteristic). This is because a rationally connected variety has trivial Chow group of zero-cycles. Explicitly let X be a smooth projective fourfold whose MRC quotient has dimension ≤ 2 , cf. [14]. This means that there is a dominant rational map $X \dashrightarrow S$ with $\dim S = 2$ and with rationally connected general fibre. Let then $f : X' \rightarrow S$ be a desingularization of the rational map $X \dashrightarrow S$. Then, a general fibre of f is rationally connected. Therefore, by theorem 1, X' has a self-dual Murre decomposition and satisfies the motivic Lefschetz conjecture.

The fourfold above is not of general type. A natural question is to ask whether it is possible to construct a fourfold of general type that has a self-dual Murre decomposition. In [20, §2.3 & Cor 4.12], we produced examples of such fourfolds. These fourfolds had the property of having trivial Chow group of zero-cycles. Obvious examples were given by the product of two surfaces of general type with trivial Chow group of zero-cycles (e.g.

Godeaux surfaces). Another example, a fourfold of Godeaux type, was given. The strategy consisted in checking the validity of the generalised Hodge conjecture for this fourfold.

We are now going to give an example of fourfold of general type with non-trivial (and in fact non-representable) Chow group of zero-cycles that has a self-dual Murre decomposition. Let's take up the notations of the previous section and let's consider the family $X \rightarrow \mathbf{P}(\bar{V})$. Let then S be a high degree (i.e. ≥ 5) complete intersection which is a smooth surface in $\mathbf{P}(\bar{V})$ meeting $\mathbf{P}(\bar{V} - V)$ transversely. Then, $X|_S$ is a projective fourfold with $X|_S \rightarrow S$ having a smooth Godeaux surface as a general fibre. A desingularization $X' \rightarrow X|_S$ gives a morphism $X' \rightarrow S$ with general fibre being of general type and having trivial Chow group of zero-cycles. This is because these two conditions are birational invariants. The high degree condition on S imposes that S is of general type. Therefore, by Viehweg's result [23] X' is of general type, and by theorem 1 X' has a self-dual Murre decomposition and satisfies the motivic Lefschetz conjecture.

10 Application to unramified cohomology

Following the fundamental result of Colliot-Thélène, Sansuc and Soulé [3] which asserts that the degree three unramified cohomology groups $H_{nr}^3(S/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$ vanish for all prime numbers l for S a smooth projective surface defined over a field k which is either finite or separably closed, it is proved in [2] that if X is a smooth projective variety defined over a field k which is either finite or separably closed such that its Chow group of zero-cycles is supported on a surface, then the groups $H_{nr}^3(X/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$ are finite for all prime numbers l and vanish for almost all l . Therefore, any variety X defined over a finite field or a separably closed field such that its restriction to a universal domain X_Ω satisfies the assumptions of theorem 1.3 has finite degree three unramified cohomology $\bigoplus_l H_{nr}^3(S/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$. In particular, the fourfold of general type of section 9 when defined over a finite field or a separably closed field has finite degree three unramified cohomology. Furthermore, as a straightforward application of theorem 1.7, we get

Proposition 10.1. *Let $f : X \rightarrow C$ be a dominant and generically smooth morphism from a smooth projective variety X to a smooth projective curve C defined over a field k which is either finite or separably closed. Assume that the general fibre Y of f_Ω is such that $CH_0(Y)_{\text{alg}}$ is representable. Then, $H_{nr}^3(X/k, \mathbf{Q}_l/\mathbf{Z}_l(2))$ is finite for all prime numbers l and vanishes for almost all l .*

Since unramified cohomology is a birational invariant for smooth projective varieties, the conclusion of the above theorem still holds for a smooth projective variety X' which is birational to the variety X of the theorem. For instance, we get finiteness of degree three unramified cohomology for threefolds which are the smooth compactification of one-dimensional families of smooth projective bielliptic surfaces defined over a finite field or a separably closed field.

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